COMBINATORICA 5 (2) (1985) 95—107

ON GENERALIZED RAMSEY NUMBERS FOR TREES

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Received 18 April 1984
Revised 9 August 1984

1. Introduction

Various constructions using resolvable block designs and Latin squares are presented to give an improved general lower bound for the Ramsey number of trees, as well as upper and lower bounds for a generalization of the concept of Ramsey number for trees. We succed in calculating the exact value for these numbers for several infinite series of parameters.

Let G be a finite graph and k a natural number. The generalized (diagonal) Ramsey number $r(G, k)$ is defined by

$r(G, k) = \mathrm{Min}\,\{r \mid$ whenever the edges of the complete graph K_r on r vertices are partitioned into k classes ("colours"), at least one of the monochromatic subgraphs of K_r contains a copy of $G\}$. For the special case that G is a tree (i.e. a connected acyclic graph), lower and upper bounds are given in [6, 7]. These bounds are valid simultaneously for all trees with a given number of edges. Let $\mathcal{T}_n$ denote the class of trees with n edges (and $n+1$ vertices). For every rational q, let $\lfloor q \rfloor$ resp. $\lceil q \rceil$ be the greatest integer smaller or equal respectively the smallest integer greater or equal q. The best general bound that is to be found in the literature ([6], without proof), seems to be $r(T_n, k) > \lfloor (k+1)/2 \rfloor (n-1)$ for every $T_n \in \mathcal{T}_n$, $k \geqq 1, n \geqq 1$. (The proof given in [7, p. 29] seems to be defective). We propose to study the following numbers:

$r(\mathcal{T}_n, k) = \mathrm{Min}\,\{r \mid$ whenever the edges of the complete graph K_r are k-colored, there is a monochromatio connected subgraph of K_r with more than n vertices$\}$
$= \mathrm{Min}\,\{r \mid$ every k-coloring of the edges of K_r produces a tree $T \in \mathcal{T}_n$ as a monochromatic subgraph of $K_r\}$

for natural numbers $n, k \geqq 2$.

These numbers can be viewed as a generalization of the concept of Ramsey numbers, the class $\mathcal{T}_n$ of all trees with n edges replacing an individual such tree. It is clear from the definition that

$$r(T, k) \geqq r(\mathcal{T}_n, k) \quad \text{for every} \quad T \in \mathcal{T}_n.$$

We use a certain type of latin squares of even order to prove the general lower

AMS subject classification (1980): 05 C 55 bound (Theorem (3.1))

$$r(\mathcal{T}_n, k) > 2\lfloor n/2\rfloor\lfloor(k+1)/2\rfloor, \quad k \geqq 2, \quad n \geqq 2,$$

thus providing also an improved lower bound for the Ramsey numbers $r(T, k)$, $T \in \mathcal{T}_n$. As a consequence we get an improvement of the lower bound of [6] for the Ramsey numbers of forests (i.e. acyclic graphs): $r(F, k) > \lfloor(k+1)/2\rfloor \lfloor\sqrt{n}\rfloor$ for every forest F with n edges, without isolated vertices.

If the edges of a graph are k-colored (i.e. partitioned in k classes) and c is one of the colours, let $G(c)$ be the subgraph of G induced by the edges with colour c. For every graph G, $E(G)$ resp. $V(G)$ denotes the set of edges resp. vertices. If X is a finite set, $K(X)$ is the complete graph with X as vertex-set. A k-coloring of the edges of a complete graph K_u will be called an (n, k)-coloring, if the connected components of $K_u(c)$ have at most n vertices for every colour c.

A good deal of our interest in the numbers $r(\mathcal{T}_n, k)$ stems from the fact, that (n, k)-colorings turn out to be closely connected with finite geometries. Resolvable balanced incomplete block designs (short: resolvable BIBD) and latin squares make their appearance. It was already mentioned that we need a certain type of latin squares of even order to derive our lower bounds for $r(\mathcal{T}_n, k)$. Section 4 is dedicated to the proof, that these latin squares always exist. In Lemma 3.4 we introduce an inductive process, a generalization of Lindström's method ([10]), which uses the existence of sets of mutually orthogonal latin squares for getting better lower bounds for certain values of n and k. It is shown in Lemma 2.1, that the existence of a resolvable BIBD with n points per block, replication k and $\lambda=1$ (necessarily then $k \equiv 1 \pmod{n}$) induces an (n, k)-coloring of $K_{k(n-1)+1}$. We have then

$$r(\mathcal{T}_n, k) \leqq k(n-1)+2.$$

Another interesting situation occurs if, for parameters (n, k), $k \equiv 1 \pmod{n}$, there is no resolvable BIBD with n points per block, replication k and $\lambda=1$. For instance, $r(\mathcal{T}_6, 7) < 37$ as there is no projective plane of order 6. We can regard the number $r(\mathcal{T}_6, 7)$ as measuring how near one can get to a projective plane of order 6. Unfortunately we have not been able to determine this number. In Section 2 we use purely combinatorial methods for deriving upper bounds. Surprisingly, these bounds seem to be quite good. The proof of (3.5) shows, that equality holds for several series of parameters (n, k). The same is true for the general lower bound. We have $r(\mathcal{T}_n, k) = 2\lfloor n/2\rfloor\lfloor(k+1)/2\rfloor+1$ in the following cases: (i) $n=2$, k arbitrary (ii) $k=2$, $n \equiv 0 \pmod 2$ (iii) $k=3$, $n=4$ or 6. In (3.5), numbers $r(\mathcal{T}_n, k)$ are calculated for several infinite series and for some individual values of (n, k).

It might be of interest to compare the number $r(\mathcal{T}_n, k)$ with the generalized Ramsey numbers $r(T, k)$, $T \in \mathcal{T}_n$. By definition $r(\mathcal{T}_n, k) \leqq \text{Min}\,\{r(T, k) \mid T \in \mathcal{T}_n\}$. The case $n=3$, which is the smallest interesting value of n, shows that equality does not hold in general. We have $\mathcal{T}_3 = \{P_3, K_{1,3}\}$, where P_3 is the path and $K_{1,3}$ is the star, both with 3 edges. If $k \equiv 1 \pmod 3$, then $r(\mathcal{T}_3, k) = r(P_3, k) = r(K_{1,3}, k) = 2k+2$, but if $k \equiv 2 \pmod 3$, then $r(\mathcal{T}_3, k) = 2k$, $r(P_3, k) = 2k+1$, $r(K_{1,3}, k) = 2k+2$ (Theorem (3.5), [3] and [9]).

2. Upper bounds

Lemma 2.1. *Let* $n \geqq 2, 1 < k \equiv 1 \pmod n$. *Then* $r(\mathcal{T}_n, k) \leqq k(n-1)+2$.

Equality holds exactly if there is a RBIBD (resolvable balanced incomplete block design) with n points per block, replication k and $\lambda = 1$ (and $k(n-1)+1$ blocks).

Proof. Consider an (n, k)-coloring of the complete graph K_u. If V is a vertex, then at most $n-1$ of the edges through V can have the same colour. This yields $u \leqq k(n-1)+1$. If equality holds, then by the same argument every connected component of $K_u(c)$ (c a colour) has n vertices and all its vertices are connected by edges with colour c. Thus the monochromatic connected components are the blocks of a block design and the coloring provides a resolution with the sets of equally colored blocks as spreads.

On the other hand it is obvious that a resolvable BIBD with the above parameters yields a (n, k)-coloring of $K_{k(n-1)+1}$. ∎

The theorem of Ray-Chaudhury and Wilson ([12]) shows, that a resolvable BIBD with blocks of cardinality $n, \lambda = 1$ and $k \equiv 1 \pmod n$ will always exist if k is large enough, i.e. if $k \geqq k_0(n)$. Further $k_0(n) = n+1$ if $n \leqq 4$ by [11, 12].

Corollary 2.2. *Let* $n \geqq 2$. *Then there is a number* $k_0(n)$ *such that* $r(\mathcal{T}_n, k) = k(n-1)+2$ *whenever* $k \geqq k_0(n)$, $k \equiv 1 \pmod n$. *Moreover,* $r(\mathcal{T}_n, k) = k(n-1)+2$ *if* $k \equiv 1 \pmod n$, $n \leqq 4$. ∎

In order to derive our general upper bounds for $r(\mathcal{T}_n, k)$, we need a combinatorial Lemma.

Definition 2.3. Let b, a be natural numbers, $0 < b \leqq a$, $[b, a] = \{n \mid n \in \mathbf{N}, b \leqq n \leqq a\}$. A natural number n is said to be $[b, a]$-*representable* if there exist $x_i \in [b, a]$, $1 \leqq i \leqq t$, such that $n = \sum_{i=1}^{t} x_i$. Let $\Phi(a, b)$ denote the set of those natural numbers which are not $[b, a]$-representable. Further set

$$v(a, b, n) = \operatorname{Max}\left\{\sum_i \binom{x_i}{2} \,\middle|\, n = \sum_i x_i,\ x_i \in [b, a]\right\}.$$

Lemma 2.4. *Let* $b \leqq a$. *A natural number n is exactly then not* $[b, a]$ *representable, if there is* $k \in \mathbf{N}$ *such that* $n \in [ka+1, (k+1)b-1]$. *If* $b < a$, *then* $k_0 = [(b-1)/(a-b)]$ *is the smallest natural number k such that the interval* $[ka+1, (k+1)b-1]$ *is empty. Further* $\Phi(a, b) = \bigcup_{i=0}^{k_0-1} [ia+1, (i+1)b-1]$ *if* $a > b$.

Proof. Let $n \in [ka+1, (k+1)b-1]$. If $n = \sum_{i=1}^{t} x_i$; $x_i \in [b, a]$, then $k < t < k+1$, contradiction. Thus n is not $[b, a]$-representable. On the other hand it is obvious that every $n \in [(k+1)b, (k+1)a]$ is a sum of $k+1$ natural numbers from the interval $[b, a]$. Now all the statements of the Lemma are obviously true.

Corollary. *Let* $\{a, b, n\} \subset \mathbf{N}$, $b \leqq a$, $n \notin \Phi(a, b)$. *Write* $n = [n/a]a + r$, *where* $0 \leqq r \leqq a-1$.

If $r=0$, *then* $v(a,b,n)=\frac{n}{a}\binom{a}{2}$.

If $r\neq 0$, *then* $v(a,b,n)=x\binom{a}{2}+\binom{c}{2}+y\binom{b}{2}$, *where* $x=\lfloor n/a\rfloor-\lceil (b-r)/(a-b)\rceil=\lfloor n/a\rfloor-\lfloor (b-r-1)/(a-b)\rfloor-1$, $y=0$ *if* $n-xa<a$, $y=\lfloor (n-(x+1)a)/b\rfloor+1$ *if* $n-xa>a$.

Proof. If $c,d\in\mathbf{N}, c\leqq d+1$, then $\binom{c}{2}+\binom{d}{2}\leqq\binom{c-1}{2}+\binom{d+1}{2}$. This shows, that a $[b,a]$-representation $n=\sum x_i$, $x_i\in[b,a]$, which yields the maximal value of $\sum\binom{x_i}{2}$, must have the form $n=xa+c+yb$, $b\leqq c<a$, and x is the largest natural number such that $n-xa\notin\Phi(a,b)$. As $\Phi(a,b)$ has been given explicitly in (2.4), the statements of the corollary follow easily. ∎

Lemma 2.5. *(i) Let* $k\equiv 0 \pmod n$. *Then* $r(\mathcal{T}_n,k)\leqq k(n-1)+1$. *(ii) Let* $k\equiv K \pmod n$, $2\leqq K<n$, *set* $\varrho=(k-K)/(k-1)$. *If* $4K>3n+\varrho-\sqrt{n(n+8-2\varrho)-\varrho(8-\varrho)}$, *then* $r(\mathcal{T}_n,k)\leqq(k-1)n-(k-K)+1$. *Otherwise* $r(\mathcal{T}_n,k)\leqq\lfloor k(n-1)+1/2-\sqrt{n(K-\varrho)-K(K-\varrho-1)-(\varrho-1/4)}\rfloor+1$.

Proof. *(1)* Consider a (n,k)-coloring of K_u. If V is a vertex, then no more than $n-1$ of its edges can have the same colour. Thus every monochromatic connected component of K_u has at least $u-(k-1)(n-1)$ edges. Set $b=u-(k-1)(n-1)$. By counting edges of K_u we get the inequality $\binom{u}{2}\leqq kv(n,b,u)$, where $v(n,b,u)$ is the number introduced in Definition 2.3.
(2) Proof of (i). Let $0<k\equiv 0 \pmod n$, $u=k(n-1)+1$. We get $b=n$, hence necessarily $u\equiv 0 \pmod n$ by Lemma 2.4, but this is not the case.
(3) Let $k\not\equiv 0,\ 1 \pmod n$, $u\equiv 0 \pmod n$. Then $u(u-1)\leqq k\frac{u}{n}n(n-1)=ku(n-1)$, $u\leqq k(n-1)+1\equiv k+1 \pmod n$. Let $k\equiv K \pmod n$, $2\leqq K<n$. Then $0<n-K+1<n$. As $u\equiv 0 \pmod n$, it follows $u\leqq k(n-1)+1-(n-K+1)=(k-1)n-(k-K)$.
(4) Let now k,K,ϱ be like in the situation of (ii), $u=(k-1)n-(k-K)+r=u_0+r$, $0<r\leqq n-1$. We have $b=K-1+r<r$. As $u\not\equiv 0 \pmod n$, we have $b<n$, thus $r<n+1-K$. Set $b'=\min\{b,n+1-K\}$. Then $v(n,b,u)\leqq v(n,b',u)=(u_0/n-1)\binom{n}{2}+\binom{n+1-K}{2}+\binom{K-1+r}{2}$. Thus

$$u(u-1)\leqq k\{(u_0-n)(n-1)+(n+1-K)(n-K)+(K-1+r)(K-2+r)\},$$

$$\begin{aligned}(k-1)r^2-r(k-1)(2n+1-2K) &\geqq u_0(u_0-1-k(n-1))+\\ &\quad +k\{n(n-1)-(n+1-K)(n-K)-(K-1)(K-2)\}\\ &=-u_0(n-K+1)+k(K-1)(2n+2-2K).\end{aligned}$$

Observe $u_0=(k-1)(n-\varrho)$. Cancel $k-1$. We get

$$\begin{aligned}r^2-r(2n+1-2K) &\geqq -(n-\varrho)(n-K+1)+(K-1)(2n+2-2K)+(1-\varrho)(2n+2-2K)\\ &=(n-K+1)(2K-\varrho-n).\end{aligned}$$

By completing the square

$$(n+1/2-K-r)^2 \geqq (n+1/2-K)^2+(n+1-K)(2K-\varrho-n)$$
$$= n(K-\varrho)-K(K-\varrho-1)-(\varrho-1/4).$$

As $r<n+1/2-K$, we get

$$r \leqq \lfloor n+1/2-K-\sqrt{n(K-\varrho)-K(K-\varrho-1)-(\varrho-1/4)}\rfloor \qquad (*)$$

It is easily seen that the right hand side of (*) is 0 if

$$4K \leqq 3n+\varrho-\sqrt{n(n+8-2\varrho)-\varrho(8-\varrho)}.$$

Assume K does not satisfy this condition. Then $K<n/2+1/4$. An easy calculation shows $r \leqq n-2(K-1)$ and this yields $b'=b$. Thus the bounds of (ii) cannot be improved by our methods. ∎

Lemma 2.6. *If* $n>2$, *then* $r(\mathcal{T}_n, n) \leqq n(n-1)$.

Proof. Assume there is an (n, n)-coloring of $K=K_{n(n-1)}$. In the terminology of (2.5) we have $b=n-1$. Let Z be a connected component of $K(c)$, c a colour. Then $|Z| \in \{n-1, n\}$.

Assume $|Z|=n-1$. Then $Z \cong K_{n-1}$. As $n(n-1)-(n-1)= =(n-1)^2 \notin \Phi(n, n-1)$, but $(n-1)^2-n \in \Phi(n, n-1)$, $K(c)$ must be a union of n disjoint copies of K_{n-1}. Let c' be a colour different from c, V a vertex of K. Then V is on $n-1$ edges of colour c'. Thus $K(c')$ is the union of $n-1$ disjoint copies of K_n. This is impossible as there is more than one such colour c'.

We have shown $|Z|=n$. Let $K(c)=\bigcup_{i=1}^{n-1} Z_i(c)$, $|Z_i(c)|=n$ be the partition of $K(c)$ in connected components, let

$$t(c) = \left|\{\overline{PQ} \mid \{P, Q\} \subseteqq V(Z_i(c)) \text{ for some } i, \overline{PQ} \text{ is not colored } c\}\right|.$$

Fix now some colour c and let Z be a connected component of $K(c')$, $c' \neq c$. If $Z \cong K_n$, then Z shares more than one vertex with some connected component of $K(c)$. Thus every connected component of $K(c')$, $c' \neq c$, yields a non-trivial contribution either to $t(c)$ or to $t(c')$. By summing up we get $\sum_{c'' \text{ colour}} t(c'' \geqq (n-1)^2$.

This leads to an improvement of the inequality used for proving Lemma 2.5: $\binom{n(n-1)}{2} \leqq n(n-1)\binom{n}{2}-(n-1)^2$, which is untrue as $n>2$. ∎

3. Lower bounds

Theorem 3.1. $r(\mathcal{T}_n, k) > 2\lfloor n/2 \rfloor \lfloor (k+1)/2 \rfloor$, $k \geqq 2$, $n \geqq 2$. The proof uses a certain type of latin square of even order.

Definition 3.2. A latin square L of order $2t$ will be called a $B(2t)$, if the following hold:

(i) L is symmetric.

(ii) Let $L=(a_{ij})$, i, $j=1,2,\dots,2t$; $L_i=\begin{pmatrix} a_{2i-1,2i-1} & a_{2i-1,2i} \\ a_{2i,2i-1} & a_{2i,2i} \end{pmatrix}$ for $i=1,2,\dots,t$. Then $L_1=L_2=\dots=L_t$ and L_1 is a latin square. ∎

We shall prove in Section 4, that $B(2t)$'s exist for every natural number t.

Proof of Theorem 3.1. Let $t=\lfloor(k+1)/2\rfloor$, $u=2t\lfloor n/2\rfloor$. We use a partition of the complete graph K_u: $K_u=\bigcup_{i=1}^{2t} A_i$, where A_i is a complete graph on $\lfloor n/2\rfloor$ vertices, and set $A'_j=A_{2j-1}\cup A_{2j}$, $j=1,2,\dots,t$. The edges belonging to one of the A'_j are all given the same colour 1, whereas edges joining vertices of A_i and A_k, $i\neq k$, $\{i,k\}\neq\{2j-1,2j\}$, $j=1,2,\dots,t$ are colored according to the entries of a $B(2t)$ (which are chosen to be different from 1). This method uses $2t-1$ colours. ∎

Lemma 3.3. *Let F be a forest (i.e. an acyclic graph) with n edges, without isolated vertices. Then*

$$r(F,k) > \lfloor(k+1)/2\rfloor\lfloor\sqrt{n}\rfloor \quad k\geqq 2, \quad n\geqq 2.$$

Proof. Erdős—Graham [6] prove a weaker lower bound. Our proof is similar to theirs. Let $t=\lfloor(k+1)/2\rfloor$, $m=\lfloor\sqrt{n}\rfloor$. If $T\subseteqq F$, $T\in\mathcal{T}_{m+1}$, we are done by Theorem 3.1. Thus we can assume, that connected components of F have at most m edges. Consider the number $u=u(F)$, the minimal cardinality of a set of vertices with the property, that every edge of F is incident with one of these vertices. We might call $u(F)$ the "blocking number" of F in analogy to the blocking number of finite incidence structures (see [2]). The elementary fact, that $r(F,k)>t(u-1)$ (see [6]) shows, that we can assume $u\leqq m$. Thus F must be the disjoint union of m trees, each with m edges and blocking number 1. Hence $F\cong mK_{1,m}$, where $K_{1,m}$ is the star with m edges. We have $n=m^2$, $m\geqq 2$. Clearly $r(F,k)\geqq r(K_{1,m},k)$. By [3] $r(K_{1,m},k)=k(m-1)+\varepsilon$, $\varepsilon\in\{1,2\}$. It remains to show $k(m-1)+\varepsilon>tm$. If $k\equiv 0 \pmod 2$, this reduces to $k(m-1)+1>km/2$, which is obviously correct. If $k\equiv 1 \pmod 2$, we get $\varepsilon=2$ by [3] and the inequality is still true. ∎

Sets of MOLS (mutually orthogonal latin squares) can be used to get better lower bounds in special situations. The following Lemma is a generalization of Lindström's construction [10], who used pairs of orthogonal latin squares for the determination of $r(P_3,k)$, $k\equiv 0 \pmod 3$, $k\neq 3^m$, $m>1$.

Lemma 3.4. *Assume $r(\mathcal{T}_n,k)>u$. If there is a set of $n-1$* MOLS *of order u, then $r(\mathcal{T}_n,u+k)>nu$.*

Proof. Write $V(K_{nu})=\bigcup_{i=1}^{n} G_i$, $|G_i|=u$. Use the same k colours for an (n,k)-coloring of G_i, $i=1,2,\dots,n$.

A set $\mathcal{M}$ of $n-1$ MOLS of order u can be represented as follows: Let $M_1,\dots,M_{n+1}$ be disjoint sets of cardinality u. Then $\mathcal{M}$ is a set of u^2 $(n+1)$-subsets of $\bigcup_{i=1}^{n+1} M_i$ with the properties:

(i) $|g\cap M_i|=1$ for every $g\in\mathcal{M}$, $i\in\{1,2,\dots,n+1\}$.

(ii) If $P \in M_i$, $Q \in M_j$, $i \neq j$, then there is exactly one $g \in \mathcal{M}$ such that $\{P, Q\} \subseteqq g$.

In our situation we identify M_i with G_i, $i=1, 2, \ldots, n$ and M_{n+1} with a set of u "new" colours. If $e=\overline{PQ}$ is an edge, $P \in G_i$, $Q \in G_j$, $i \neq j$, let g be the unique element of $\mathcal{M}$ containing P and Q and colour $\overline{PQ}$ with $g \cap M_{n+1}$. ∎

We are now in a position to determine several series of our numbers $r(\mathcal{T}_n, k)$.

Theorem 3.5.

(i) $r(\mathcal{T}_2, k)=k+2$ *if* k *is odd,* $r(\mathcal{T}_2, k)=k+1$ *if is even.*

(ii) $r(\mathcal{T}_3, k)=2k+2$ if $k \equiv 1 \pmod 3$, $r(\mathcal{T}_3, k)=2k$ *if* $k \equiv 2 \pmod 3$.
$r(\mathcal{T}_3, 3)=6$.

(iii) $r(\mathcal{T}_n, 2)=n+1$, $n \geqq 2$.

(iv) $r(\mathcal{T}_4, 3)=9$, $r(\mathcal{T}_4, 4)=11$, $r(\mathcal{T}_4 5), =r(\mathcal{T}_4, 6)=17$, $r(\mathcal{T}_4, 9)=29$.

(v) $r\left(\mathcal{T}_4, 6+16 \sum_{j=0}^{i} 4^j\right)=16 \cdot 4^{i+1}+1$, $i \geqq 0$.

$r\left(\mathcal{T}_4, 10+28 \sum_{j=0}^{i} 4^j\right)=28 \cdot 4^{i+1}+1$, $i \geqq 0$.

(vi) $r(\mathcal{T}_5, 3)=10$

(vii) $r(\mathcal{T}_6, 3)=13$

(viii) $r\left(\mathcal{T}_6, 3+12 \sum_{j=0}^{i} 6^j\right)=12 \cdot 6^{i+1}+1$, $i \geqq 0$.

(ix) $r(\mathcal{T}_8, 3)=17$.

Proof. The values of $r(\mathcal{T}_n, k)$, $n \leqq 4$, $k \equiv 1 \pmod n$ follow from Corollary 2.2.

(iii) Trivially $r(\mathcal{T}_n, 2)>n$. For an $(n, 2)$-coloring of K_{n+1} we have $b=2$. Let C be a connected component of K_{n+1}(red). As $|C|>1$ and every edge connecting a vertex of C and a vertex of its complement is blue, K_{n+1} (blue) is connected, contradiction.

(i), (ii). Let k be even. Then $r(\mathcal{T}_2, k) \geqq r(\mathcal{T}_2, k-1)=k+1$. Let $k \equiv 2 \pmod 3$. Then $r(\mathcal{T}_3, k) \geqq r(\mathcal{T}_3, k-1)=2k$. These bounds coincide with the upper bounds following from Lemma 2.5. $r(\mathcal{T}_3, 3) \leqq 6$ by Lemma 2.6. The following is a $(3, 3)$-coloring of $K(\{1, 2, 3, 4, 5\})$: red: $K(\{1, 2, 3\}) \cup \{(4, 5)\}$, blue: $\{(1, 4), (2, 4)\} \cup \{(3, 5)\}$, green: $\{(1, 5), (2, 5)\} \cup \{(3, 4)\}$.

(iv) Lemma 2.5 yields $r(\mathcal{T}_4, 3) \leqq 9$, $r(\mathcal{T}_4, 4) \leqq 12$, $r(\mathcal{T}_4, 6) \leqq 17$. As $r(\mathcal{T}_4, 6) \geqq r(\mathcal{T}_4, 5)=17$, we have $r(\mathcal{T}_4, 6)=17$. The following is a $(4, 3)$-coloring of $K(\{1, 2, \ldots, 8\})$:

red: $\{(1,2),(1,3),(2,4),(3,4)\}\cup K(\{5,6,7,8\})$.
blue: $\{(1,4),(1,5),(1,6),(4,5),(4,6)\}\cup\{(2,3),(2,7),(2,8),(3,7),(3,8)\}$.
green: $\{(1,7),(1,8),(4,7),(4,8)\}\cup\{(2,5),(2,6),(3,5),(3,6)\}$.

This shows $r(\mathscr{T}_4, 3)=9$.

The following (4, 4)-coloring of $K(\{1,2,\dots,10\})$ shows $r(\mathscr{T}_4, 4)>10$

red: $K(\{1,2,3,4\})\cup K(\{5,6,7,8\})\cup\{9,10)\}$.
blue: $K(\{1,5,9\})\cup K(\{2,6,10\})\cup\{(3,7),(3,8),(4,7),(4,8)\}$.
green: $\{(3,9),(4,9),(6,9),(3,6),(4,6)\}\cup\{(2,5)\}\cup\{(1,10),(7,10),(8,10),(1,7),(1,8)\}$.
pink: $\{(2,9),(7,9),(8,9),(2,7),(2,8)\}\cup\{(1,6)\}\cup\{(3,10),(4,10),(5,10),(3,5),(4,5)\}$.

Assume $K=K_{11}$ is (4, 4)-colored. At most 15 edges have the same colour. Assume there are 15 red edges. Then $K(\text{red})=K_4\cup K_4\cup K_3$. Thus there are no monochromatic K_4 of colours different from red. If c is a colour different from red, then at most 13 edges have colour c. We get the contradiction $55=\binom{11}{2}\leqq 15+3\times 13=54$. It follows, that there are 14 red, blue and green edges, 13 pink edges. Thus $K(\text{red})$ arises from $K_4\cup K_4\cup K_3$ by deleting an edge. This shows that there is at most one monochromatic K_4, which is not red. Thus it is impossible, that there are 14 blue and 14 green edges. We have $r(\mathscr{T}_4, 4)=11$.

(vi) The following (5, 3)-coloring of $K(\{1,2,\dots,9\})$ shows that $r(\mathscr{T}_5, 3)>9$.
red: $\{(1,2),(1,3),(2,3),(2,4),(3,4)\}\cup K(\{5,6,7,8,9\})$.
blue: $\{(1,4),(1,5),(1,6),(4,5),(4,6)\}\cup\{(2,7),(2,8),(2,9),(3,7),(3,8),(3,9)\}$.
green: $\{(2,5),(3,5),(2,6),(3,6)\}\cup\{(1,7),(1,8),(1,9),(4,7),(4,8),(4,9)\}$.

Assume $K=K_{10}$ is (5, 3)-colored. If c is a colour, V a vertex, let $K_V(c)$ denote the set of vertices Q such that (P,Q) has colour c. Assume $|K_V(\text{red})|\leqq 2$. Then without restriction $|K_V(\text{blue})|=4$, $|K_V(\text{green})|=3$. At most one of the points of $K_V(\text{blue})$ can be connected with points of K (green) by blue or green edges. Thus we get a red connected component of cardinality at least 6, contradiction. We have $|K_V(c)|\geqq 3$ for every vertex V and colour c. It follows, that monochromatic connected components all have cardinality 5, and there are $2\lceil 5\times 3/2\rceil=16$ edges of every given colour. Thus the number of edges exceeds 45, contradiction. We have $r(\mathscr{T}_5, 3)=10$.

(vii) Upper and lower bounds of Lemma 2.5 resp. Theorem 3.1 coincide.

(ix) $r(\mathscr{T}_8, 3)>16$ by Theorem 3.1. Assume $K=K(\{1,2,\dots,17\})$ is (8, 3)-colored. We have $b=3, v(8,3,17)=46$ in the notation of Definition 2.3. The standard inequality reads $|E(K)|=136\leqq 3\times 44=138$. Thus for at least one colour c we must have $K(c)=K_8\cup K_6\cup K_3$. This shows, that complete graphs K_8 and K_6 cannot be connected components of $K(c'), c'\neq c$. Obviously we get a contradiction.

(v), (viii). As $r(\mathscr{T}_4, 6)>16$, $r(\mathscr{T}_4, 10)\geqq r(\mathscr{T}_4, 9)>28$, $r(\mathscr{T}_6, 3)>12$, the lower bounds of (v) and (viii) follow by repeated application of Lemma 3.3. We only need the existence of 3 MOLS of orders 16×4^i and 28×4^i, $i\geqq 0$ resp. of 5

MOLS of orders 12×6^i, $i\geqq 0$. Sets of 5 MOLS of order 12 were constructed in [1, 5] (see [4, p. 479—480]). The statement now follows from the simple fact, that the existence of r MOLS of orders n_1 and n_2 guarantees the existence of r MOLS of order $n_1 n_2$ ([4, p. 389]). In all these cases the lower bound coincides with the upper bound of Lemma 2.5.

4. A class of latin squares

We recall the definition given in (3.2). A symmetric latin square $L=(a_{i,j})$ $i, j=1, 2, \ldots, 2t$ is called a $B(2t)$, if the submatrices $L_i=\begin{pmatrix} a_{2i-1,2i-1} & a_{2i-1,2i} \\ a_{2i,2i-1} & a_{2i,2i} \end{pmatrix}$, $i=1, 2, \ldots, t$ are latin subsquares and $L_1=L_2=\ldots=L_t$.

The problem, if $B(2t)$'s exist, is related to a problem posed by L. Fuchs ([4, p. 56]), which was studied recently by K. Heinrich ([8]).

Theorem 4.1. *A $B(n)$ exists for every even natural number n.*

Proof. We use the abbreviation "LS(n)" for "latin square of order n".

(1) Let A be a symmetric LS(a). Assume A is unipotent, i.e. with constant main diagonal. Then we get a $B(2a)$ by "blowing up", i.e. every entry i in A is replaced by U_i, where U_i is a LS(2) and there is no common entry in U_i and U_j for $i\neq j$.

(2) Exactly then does a symmetric unipotent LS(a) exist if $a=1$ or a is even ([4, p. 31]). Thus a $B(n)$ exists if $n=2$ or $n\equiv 0 \pmod 4$.

(3) **Definition.** Let L be an LS(n) and T a transversal of L. Then T determines a *permutation* $\pi(T)$ of the set $\{1, 2, \ldots, n\}$ by $\pi(T)$: $i\to j$ whenever an element of T is in position (i, j) in L. Thus $\pi(T)$ is determined by the positions of the elements of T.

(4) Let T be a transversal of L, where L is a symmetric LS(n). Then the transpose T' of T is also a transversal and $\pi(T')=\pi(T)^{-1}$. If L is unipotent, then we have in addition

(i)

$|T\cap T'|=1$ and $T\cap T'$ is on the main diagonal.

(ii)

$\pi(T)$ has exactly one fixed point.

(5) (modified prolongation). This is a modification of the method of "prolongation" as described in [4, pp. 39—40]. Let $4\leqq a\equiv 0 \pmod 4$ and A a $B(a)$. Choose notation such that $A_i=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $i=1, 2, \ldots, a/2$ (see (1)).

Assume we have a transversal T of A. We form an incomplete symmetric LS(n) $n=a+2$:

$$L=\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}.$$

Here L_1 arises from A by deleting the entries of T and T', with the exception of entries 1 and 2. The columns of L_2 are the entries of T and T' (in this order), L_3

is the transpose of L_2. Finally $L_4=A_1$. Then every entry of A occurs exactly n times in the symmetric incomplete LS(n) L.

(6) Let a, n, A, T, L be as in (5). Exactly then can L be completed to a $B(n)$, if $\pi(T)$ possesses exactly one cycle of odd length besides the fixed point, and the coordinates of the cell with entry 2 in T occur in this cycle.

Proof. Let x, y be symbols, which do not occur in A. It follows from (5), that we have to distribute x and y on the empty cells of L, so that symmetry is preserved and every row and column contains one entry x and one entry y.

First let $z=(z_1, z_2, \dots, z_i)$ be a cycle of $\pi(T)$ such that entries 1 and 2 do not occur in the cells determined by z. Consider the subsquare $L(z)$ of L determined by the rows and columns corresponding to z. Renumber the rows and columns of $L(z)$ by mapping z_j onto j. The empty cells of $L(z)$ are in positions $(1, i)$, $(j+1, j)$, $1 \leqq j \leqq i-1$ and their images under transposition. Assume $L(z)$ is completed in the required way. Then x appears equally often above and beneath the diagonal, but not on it. As x occurs exactly once in every row, it follows $i \equiv 0 \pmod 2$. Assume now $i \equiv 0 \pmod 2$. We get a completion of $L(z)$ by assigning entry x to positions $(1, i), (i, 1)$ and $(2j, 2j+1), (2j+1, 2j)$, $j=1, 2, \dots, i/2-1$. Let now $z= =(z_1, \dots, z_i)$ be the cycle of $\pi(T)$, which determines the cell with entry 2 of T. Choose notation, so that the coordinates of this cell are $(3, 4)$. Hence $z=(3, 4, \dots, z_i)$. Consider the subsquare L^* of L corresponding to rows and columns with indices $1, z_1, z_2, \dots, z_i, a+1, a+2$. We can choose x to occupy cells $(1, a+1), (a+1, 1)$. Then necessarily y occurs in $(1, a+2), (a+2, 1), (3, a+1), (a+1, 3)$ and x in $(4, a+2)$, $(a+2, 4)$. Like above we renumber rows and columns of $L(z)$ by mapping $z_j \to j$, $j=1, 2, \dots, i$. The empty cells of $L(z)$ are $(j, j+1), (j+1, j), j=3, \dots, i-1$, whereas $(1, i)$ and $(i, 1)$ have entry x, and y occupies $(2, 3)$ and $(3, 2)$. It follows, that $(2j, 2j+1)$, $(2j+1, 2j)$ have entries y, $j<i/2$. If $i \equiv 0 \pmod 2$, then $(i-2, i-1)$ has entry y, hence x occupies $(i-1, i)$. This conflicts with entry x in $(1, i)$. If $i \equiv 1 \pmod 2$, we get a completion of $L(z)$ and L^*. It is immediate, that completions of the subsquares considered here lead to a symmetric LS(n), hence to a $B(n)$. ∎

(7) **Corollary.** *Let $a \equiv 0 \pmod 4$. Assume there is a $B(a)$ possessing a transversal T, such that $\pi(T)$ has exactly one cycle of odd length besides its fixed point, and entry 2 (in the notation of (5)) occurs in a cell of T determined by this cycle. Then there is a $B(a+2)$.* ∎

(8) **Examples.** The cells belonging to transversal T of A are circled.

(i) $a = 4$,

$$A = \begin{pmatrix} ① & 2 & 3 & 4 \\ 2 & 1 & ④ & 3 \\ 3 & 4 & 1 & ② \\ 4 & ③ & 2 & 1 \end{pmatrix}, \quad \pi(T) = (1)(2, 3, 4), \quad L = \begin{pmatrix} 1 & 2 & 3 & 4 & x & y \\ 2 & 1 & x & y & 4 & 3 \\ 3 & x & 1 & 2 & y & 4 \\ 4 & y & 2 & 1 & 3 & x \\ x & 4 & y & 3 & 1 & 2 \\ y & 3 & 4 & x & 2 & 1 \end{pmatrix}$$

(ii) $a = 8$,

$$A = \begin{pmatrix} \textcircled{1} & 2 & 5 & 6 & 7 & 8 & 3 & 4 \\ 2 & 1 & 6 & 5 & \textcircled{8} & 7 & 4 & 3 \\ 5 & 6 & 1 & \textcircled{2} & 3 & 4 & 7 & 8 \\ 6 & 5 & 2 & 1 & 4 & 3 & 8 & \textcircled{7} \\ 7 & 8 & 3 & 4 & 1 & 2 & \textcircled{5} & 6 \\ 8 & 7 & \textcircled{4} & 3 & 2 & 1 & 6 & 5 \\ 3 & 4 & 7 & 8 & 5 & \textcircled{6} & 1 & 2 \\ 4 & \textcircled{3} & 8 & 7 & 6 & 5 & 2 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 2 & 5 & 6 & 7 & 8 & 3 & 4 & x & y \\ 2 & 1 & 6 & 5 & y & 7 & 4 & x & 8 & 3 \\ 5 & 6 & 1 & 2 & 3 & x & 7 & 8 & y & 4 \\ 6 & 5 & 2 & 1 & 4 & 3 & 8 & y & 7 & x \\ 7 & y & 3 & 4 & 1 & 2 & x & 6 & 5 & 8 \\ 8 & 7 & x & 3 & 2 & 1 & y & 5 & 4 & 6 \\ 3 & 4 & 7 & 8 & x & y & 1 & 2 & 6 & 5 \\ 4 & x & 8 & y & 6 & 5 & 2 & 1 & 3 & 7 \\ x & 8 & y & 7 & 5 & 4 & 6 & 3 & 1 & 2 \\ y & 3 & 4 & x & 8 & 6 & 5 & 7 & 2 & 1 \end{pmatrix}$$

$\pi(T) = (1)(2, 5, 7, 6, 3, 4, 8)$.

(9) It remains to construct a $B(n)$ for $n=4z+2$, $z\geqq 3$. Start with B, a symmetric LS$(2z)$, which arises out of the Cayley-table of the cyclic group of order $2z-1$ by prolongation with the main diagonal. The latin square B is best described as follows: consider disjoint sets $\{r_i\}$, $\{c_i\}$, $\{e_i\}$, where in each case $i\in M=\{0, 1, \dots, 2z-2\}\cup\{\infty\}$. Then $B=\{\{r_i, c_j, e_{i+j}\}|i, j\in M\}$. Here $i+j$ is the ordinary sum modulo $2z-1$ if $i\neq\infty\neq j$ and $i\neq j$, whereas $i+\infty=\infty+i=2i$ (mod $2z-1$) and $i+i=\infty+\infty=\infty$. Then B possesses an isomorphism φ of order $2z-1$: φ: $r_i\to r_{i+1}, c_i\to c_{i+1}, e_i\to e_{i+2}, i\neq\infty$ (indices modulo $2z-1$), φ fixes r_∞, c_∞ and e_∞.

Finally A is the $B(4z)$, which arises by "blowing up" B. We write $A=\alpha(B)$, where $\alpha(\{r_i, c_j, e_{i+j}\})=A_{ij}=$

$$\Big\{\{r'_{\alpha(i)+1}, c'_{\alpha(j)+1}, e'_{\alpha(i+j)+1}\}, \{r'_{\alpha(i)+1}, c'_{\alpha(j)+2}, e'_{\alpha(i+j)+2}\}, \{r'_{\alpha(i)+2}, c'_{\alpha(j)+2}, e'_{\alpha(i+j)+1}\}, \{r'_{\alpha(i)+2}, c'_{\alpha(j)+2}, e'_{\alpha(i+j)+1}\}\Big\}$$

for $i, j\in M$, where $\alpha(i)=2i\in\mathbf{Z}$ for $i\in M-\{\infty\}$, and $\alpha(\infty)=4z-2$. Thus rows, columns and entries of A are numbered 1 through $4z$. We have to construct a transversal T of A, which satisfies the conditions of (7). Observe, that α induces a mapping α^{-1}: $A\to B$ by $\alpha^{-1}(\{r'_u, c'_v, e'_w\})=\{r_i, c_j, e_{i+j}\}$ iff $\{r'_u, c'_v, e'_w\}\in A_{ij}$. Assume T is a transversal of A. Then $D=\alpha^{-1}(T)$ has the property, that every row and column of B contains exactly two elements of D and every entry occurs exactly twice in cells of D. We might call D a "double transversal". Our construction starts by choosing D to be the union of two disjoint transversals of B. Set $T_1=\{\{r_0, c_0, e_\infty\}, \{r_1, c_{2z-2}, e_0\}, \{r_\infty, c_{z-1}, e_{2z-2}\}, \{r_z, c_\infty, e_1\}\}\cup\{\{r_i, c_{i-1}, e_{2i-1}\}|i=2, 3, \dots, 2z-1, i\neq z\}$, and $T_2=T_1^{\varphi^2}$. As $z\geqq 3$, T_1 and T_2 are disjoint transversals of B. We set $D=T_1\cup T_2$. Write $T_k(i)=\{r_i, c_j, e_{i+j}\}$ if the given cell is in T_k, $k=1, 2$. The transversal T is to be chosen in $\alpha(D)$, i.e. for every element $d\in D$ we have to choose one of the four cells of $\alpha(d)$, so that we get a transversal satisfying the conditions of (7). The construction is given in the Table. The choice of the element from $\alpha(d)$ is indicated by using the pairs (1, 1), (1, 2), (2, 1) or (2, 2), which denote the position in the 2×2-matrix $\alpha(d)$. It is easy to check, that the set T is a transversal of A.

(*i*) $z = 1 \pmod 2$, $z \geqq 5$

(1,1)	(1,2)	(2,1)	(2,2)
$T_1(0)$	$T_1(2)$	$T_1(1)$	$T_1(z)$
$T_1\left(2i \mid i = \frac{z+3}{2}, \ldots, z-1\right)$	$T_1\left(2i+1 \mid i = 1, \ldots, \frac{z-3}{2}\right)$	$T_1(z+1)$	$T_1(z+2)$
$T_1\left(2i \mid i = 2, \ldots, \frac{z-1}{2}\right)$	$T_1(\infty)$	$T_1\left(2i+1 \mid i = \frac{z+3}{2}, \ldots, z-2\right)$	
$T_2(z+1)$	$T_2(1)$	$T_2(0)$	$T_2(\infty)$
$T_2(z)$	$T_2\left(2i+1 \mid i = \frac{z+1}{2}, \ldots, z-2\right)$	$T_2(2)$	$T_2\left(2i \mid i = \frac{z+3}{2}, \ldots, z-1\right)$
		$T_2\left(2i+1 \mid i = 1, \ldots, \frac{z-3}{2}\right)$	$T_2\left(2i \mid i = 2, \ldots, \frac{z-1}{2}\right)$

(*ii*) $z = 0 \pmod 2$, $z \geqq 4$

(1,1)	(1,2)	(2,1)	(2,2)
$T_1(0)$	$T_1(2)$	$T_1(\infty)$	$T_1(1)$
$T_1(z)$	$T_1\left(2i+1 \mid i = 1, \ldots, \frac{z}{2}-1\right)$		
$T_1\left(2i \mid i = \frac{z}{2}+1, \ldots, z-1\right)$	$T_1(z+1)$		
$T_1\left(2i \mid i = 2, \ldots, \frac{z}{2}-1\right)$	$T_1\left(2i+1 \mid i = \frac{z}{2}+1, \ldots, z-2\right)$		
$T_2(\infty)$	$T_2(1)$	$T_2(0)$	$T_2(z+1)$
		$T_2(2)$	$T_2(z)$
		$T_2\left(2i+1 \mid i = 1, \ldots, \frac{z}{2}-1\right)$	$T_2\left(2i \mid i = \frac{z}{2}+1, \ldots, z-1\right)$
		$T_2\left(2i+1 \mid i = \frac{z}{2}+1, \ldots, z-2\right)$	$T_2\left(2i \mid i = 2, \ldots, \frac{z}{2}-1\right)$

(*iii*) $z = 3$

(1,1)	(1,2)	(2,1)	(2,2)
$T_1(0)$	$T_1(2)$	$T_1(1)$	$T_2(\infty)$
$T_1(3)$	$T_1(\infty)$	$T_2(0)$	$T_2(4)$
$T_2(1)$	$T_1(4)$	$T_2(2)$	$T_2(3)$

Table

Finally we have to determine the permutation $\pi(T)$ as defined in (3).

(i) Let $5 \leqq z \equiv 1 \pmod 2$. Then $\pi(T) = (1)(2, 3, 8, 9, 7, 6, 5, 4, 4z-3, 4z-5, 4z-2)(2z, 2z+2, 4z, 2z+6, 2z+4, 2z+1, 2z+3, 4z-1) \prod_{i=2}^{(z-3)/2} (4i+2, 4i+4, 4i+5, 4i+3) \prod_{i=(z+1)/2}^{z-3} (4i+3, 4i+6, 4i+8, 4i+5)$.

(ii) If $6 \leqq z \equiv 0 \pmod 2$, then $\pi(T) = (1)(2, 3, 8, 9, 7, 6, 5, 4, 4z-2)(2z-2, 2z, 2z+1, 4z-1, 2z+5, 2z+3, 2z+2, 2z+4, 4z, 2z-1) \prod_{i=2}^{z/2-2} (4i+2, 4i+4, 4i+5, 4i+3) \prod_{i=z/2+1}^{z-2} (4i+2, 4i+4, 4i+5, 4i+3)$. For $z=4$ we have $\pi(T) = (1)(2, 3, 8, 9, 15, 13, 11, 10, 12, 16, 7, 6, 5, 4, 14)$. The construction for $z=3$ (i.e. $n=14$) is given in the Table. We get $\pi(T) = (1)(2, 3, 7, 11, 6, 5, 4, 9, 8, 10, 12)$

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